M. Sc. (First Semester) 2013

Model Ans.

Mathematical Physics.

1.

(i) c	(vi) c
(ii) a	(vii) c
(iii) c	(viii) b
(iv) c	(ix) c
(v) c	(x) a

2.

2.

$$\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} \quad \frac{1}{z\left(1-\frac{1}{z}\right)}$$

As $1 < |z| < 2$ Therefore,

$$\frac{z}{2} < 1 \quad \& \quad \frac{1}{z} < 1$$

$$f(z) = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$-\frac{1}{z}\left(1+z^{-1}+z^{-2}+z^{-3}+\cdots\right)$$

$$= \cdots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{1}{16}z^3$$

3.

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

$z = e^{i\theta}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$ | $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$

$dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C \frac{dz/iz}{3-2(z+z^{-1})/2+(z-z^{-1})/2i}$$

$$= \oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i}$$

where C is the circle of unit radius with centre at the origin.

The poles of $\dfrac{2}{(1-2i)z^2+6iz-1-2i}$ are simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2-4(1-2i)(-1-2i)}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)} = 2-i,\ 2-i/5$$

only $(2-i)/5$ lies inside C.

Residue at $(2-i)/5 = \lim_{z\to(2-i)/5} \left\{z-(2-i)/5\right\}\left\{\dfrac{2}{(1-2i)z^2+6iz-1-2i}\right\}$

$$= \frac{1}{2i}$$

Thus $\displaystyle\oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right)$

$$= \pi$$

4. Let,

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= 3(-3+2) - 1(2+1) + 2(4+3)$$

$$= 8$$

The given Eq^n can be expressed like $AX = C$

where, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $C = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

and $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$\therefore X = CA^{-1}$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

Cofactor of (1.1) element $= (-1)^{1+1} \begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} = -1 = A_{11}$

Similarly $A_{12} = -3$

$A_{13} = 7$

$B_{21} = 3$; B_{22} 1 ; $B_{23} = -5$

$C_{31} = 5$; $C_{32} = 7$; $C_{33} = -11$

$$\therefore \text{Adj } A = \begin{vmatrix} A_{11} & B_{21} & C_{31} \\ A_{12} & B_{22} & C_{32} \\ A_{13} & B_{23} & C_{33} \end{vmatrix} = \begin{vmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{vmatrix}$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

Therefore, from equation (1),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{x=1, \quad y=2, \quad z=-1}$$

5. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3 - 20\lambda + 8 = 0$

By Cayley-Hamilton theorem, $A^3 - 20A + 8I = 0$

where $A = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix}$

find $A \cdot A = A^2$

then find $A^3 =$

~~Satisf~~ Put A^3 and A in $A^3 - 20A + 8I$

value will be zero

Thus Cayley-Hamilton theorem verified.

Now,

$$A^{-1} = \frac{5}{2} I - \frac{1}{8} A^2$$

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

6. $$y'' - 2xy' + 2\alpha y = 0$$

$$y = \sum_{l=0}^{\alpha} a_l x^{k+l}$$

$$y' = \sum a_l (k+l) x^{(k+l-1)}$$

$$y'' = \sum a_l (k+l)(k+l-1) x^{k+l-2}$$

$$\therefore \sum a_l (k+l)(k+l-1) x^{k+l-2} - \sum 2 a_l (k+l-\alpha) x^{k+l} = 0 \quad \text{--- (1)}$$

Equating the co-effen of the first term (i.e x^{k-2})

by putting $l = 0$

$$a_0 k(k-1) = 0$$

if $a_0 \neq 0$, $k = 0, 1$

Eqt the co-effen of x^{k-1} to 0, $a_1 (k(k+1)) = 0$

when $k = 0$ $\quad$ when $k = 1$

$a_1 \neq 0$ $\quad$ $a_1 = 0$

eqt the co-effen of x^{k+j} in Eq (1)

we have $$a_{j+2} = \frac{2(k+j) - 2\alpha}{(k+j+2)(k+j+1)} a_j$$

Case I, for $k = 0$
$$a_{j+2} = \frac{2j - 2\alpha}{(j+2)(j+1)} a_j$$

$$a_2 = \frac{-2\alpha}{2.1} a_0 \qquad a_3 = -\frac{2(\alpha-1)}{3!} a_1$$

$$a_4 = +\frac{2^2 \alpha(\alpha-2)}{4!} a_0 \qquad a_5 = \frac{2^2 (\alpha-1)(\alpha-3)}{5!} a_1$$

$$a_{2r} = \frac{(-1)^r 2^r \alpha(\alpha-2)\cdots(\alpha-2r+2)}{(2r)!} \; ; \; a_{2r+1} = \frac{(-2)^r (\alpha-1)(\alpha-3)\cdots(\alpha-2r+1)}{(2r+1)!}$$

$\therefore y_1 = \sum_{\ell} a_\ell x^\ell$

$$= a_0 \left[1 + \sum_{\ell=1}^{\alpha} \frac{(-1)^\ell 2^\ell}{(2\ell)!} \alpha(\alpha-2)\cdots(\alpha-2\ell+2)\, x^{2\ell}\right]$$

$$+ a_1\left[x + \sum_{\ell=1}^{\alpha} \frac{(-1)^\ell 2^\ell}{(2\ell+1)!}(\alpha-1)(\alpha-3)\cdots(\alpha-2r+2)\, x^{2r+1}\right]$$

—(2)

Case II for $n=1$

$$a_{\ell+2} = \frac{2(1+\ell-\alpha)}{(\ell+3)(\ell-2)} a_\ell$$

$$a_2 = -\frac{2(\alpha-1)}{3!} a_0 \qquad a_{2r} = \frac{(-1)^r 2^2(\alpha-1)(\alpha-3)\cdots(\alpha-2r+1)}{(2r+1)!}$$

$$a_4 = \frac{2^2(\alpha-1)(\alpha-3)}{5!} a_0$$

$$y_2 = a_0 x\left[1 - \frac{2(\alpha-1)}{3!}x^2 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^4 \cdots\right]$$

$$+ (-1)^r \frac{2^r(\alpha-1)(\alpha-3)\cdots(\alpha-2r+1)}{2r+1} x^{2r}.$$

—(3)

y_2 ~~$y_1 A B_2$~~ Here the sol^n (2) is the second part of (3) so, we h to get the independent sol^n we have to chorse costant a_0 or $a_1 = 0$ in two Eq^n.

7) (a) $\phi_n(x) = e^{-x/2} L_n(x)$

$$\int_0^\infty \phi_m(x)\,\phi_n(x)\,dx = \int_0^\infty e^{-x} L_n(x)\, L_m(x)\,dx$$

$$= \delta_{mn}$$

Rodrigue formula is $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$

$$\int_0^\infty e^{-x} x^m L_n(x) = \int_0^\infty \frac{x^m}{n!}\,\frac{d^n}{dx^2}\left(x^n e^{-x}\right)$$

$$= \dots\dots$$

$$= 0 \quad \text{if } n > m$$

Similarly, $\int_0^\infty e^{-x} x^n L_m(x)\,dx = 0$ if $m > n$

$$\therefore \int_0^\infty e^{-x} L_m(x)\, L_n(x)\,dx = 0, \quad \text{for } m \neq n$$

for $\underline{\underline{m=n}}$
$$\int_0^\infty e^{-x}\{L_n(x)\}^2\,dx = \frac{(-1)^n}{n!}\int_0^\infty e^{-x} x^n L_n(x)\,dx$$

$$\int_0^\infty e^{-x}\{L_n(x)\}^2 = \frac{(-1)^n}{n!}\int_0^\infty e^{-x} x^n \left[\frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)\right]dx$$

$$= \frac{(-1)^n}{n!}\int_0^\infty e^{-x}\,\frac{e^x}{n!}\,x^n\,\frac{(-1)^n}{x^n}\,\frac{n!}{1}\,\frac{d^{n-n}}{dx^{n-n}}\left(x^n e^{-x}\right)dx$$

$$= \dots$$

$$= \frac{1}{n!}\int_0^\infty x^n e^{-x}\,dx = 1$$

(b)

$L_n(x) = (-1)^n\left[x^n - \frac{n^2}{1!}x^{n-1} + \frac{n^2(n-1)^2}{2!}x^{n-2} - \dots (-1)^n n!\right]$

$L_1(x) = -1[x-1]$

$= 1-x.$

8.

Let $f(x) = e^{-ax}/x$, then its Foiurier Sine transform

i.e. $F_s\{f(x)\} = \int_0^\alpha f(x) \sin sx \, dx$

$$= \int_0^\alpha \frac{e^{-ax}}{x} \sin sx \, dx$$

$$= F(s), \text{ Say}$$

Differenciating both sides w.r.t. s, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\alpha \frac{x e^{-ax} \cos sx}{x} \, dx$$

$$= \int_0^\alpha e^{-ax} \cos sx \, dx$$

$$= \frac{a}{s^2+a^2}$$

Integrating w.r.t. s, we obtain

$$F(s) = \int \frac{a}{s^2+a^2} \, ds$$

$$= \tan^{-1}\frac{s}{a} + c$$

But $F(s) = 0$, when $s = 0$

$$\therefore c = 0$$

Hence $F(s) = \tan^{-1}(s/a)$